Contributed paper

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A super-resolution method based on signal fragmentation

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General idea of the super-resolution method based on the signal-spectrum extrapolation beyond transmission-channel band has been presented. Main limitations of former realizations of the method have been discussed. A new realization procedure of the method, removing those limitations, has been proposed. It consists in signal fragmentation before transmission through a low-pass channel. Results of a few numerical experiments done by means of original computing tools has been presented and discussed.

Keywords: signal restoration, super-resolution, signal processing.

1. Introduction

A lot of super-resolution methods of signal restoration have been proposed [1]. The approach presented in this paper belongs to the methods which allow improving resolution of signal by recovering spectrum beyond available band. Within the confines of this approach, restoration of diffraction-limited signal by means of spectrum extrapolation is known. Harris [2] showed that there are not two different objects of finite size with the same Fourier spectrum. Therefore, he concluded that object details could be restored by variety of techniques. Barnes [3] proposed the method of object restoration in one-dimensional, diffraction-limited imaging system. Extension of the method to two-dimensional, diffraction-limited imaging system has been presented by Frieden [4]. He described the method of spectrum extrapolation based on a set of prolate spheroidal wave functions [5,6]. Finally, Rusforth and Harris [7] considered the method in the presence of noise. They showed also some limitations of this method. Mathematical and physical background of super-resolution relating to digital images has been presented in Ref. 8, whereas the details of the methodology of computing spheroidal wave functions have been shown in Ref. 9. Bertero and De Mol [1] analysed all known super-resolution methods and concluded that super-resolution, in the sense of out-of-band extrapolation, is feasible only in the case where the size of objects is not too large in comparison with the resolution limit of the imaging system. This property significantly limits practical applications of the method. The main aim of this paper is to propose the method removing this limitation, and to give computing tools for practical realization of the method.

2. General idea of super-resolution methods

In the classical approach, band-limited transmission channel results in limited resolution of the transmitted signal. Out-of-band extrapolation of the signal spectrum without additional information is impossible. Cutting the signal domain before the band-limited transmission is the main idea of the super-resolution methods. It generates additional information in the transmitted band. That information is necessary for out-of-band spectrum extrapolation.

In Fig. 1, two similar signals of only one frequency ω_0, higher then the cut-off frequency $\Omega/2$ of the band-limited transmission channel, are presented. The first one has unlimited domain, whereas the second one has T-limited domain. Respective spectra of the signals are shown on the right-hand side of Fig. 1. The spectrum of the unlim-

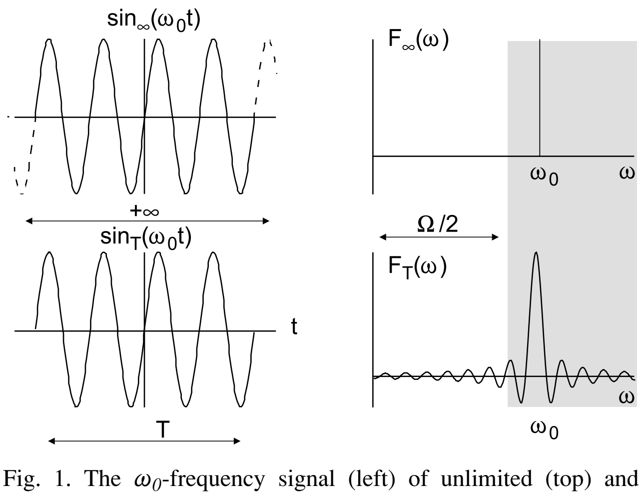

Fig. 1. The ω_0-frequency signal (left) of unlimited (top) and T-limited (bottom) domain, and its spectrums (right); dark field denotes frequencies filtered by the transmission channel with cut-off frequency $\Omega/2$.

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ited-domain signal is represented by the Dirac delta function, whereas the spectrum of the limited-domain signal takes infinite band. After band-limited transmission process, the first signal is completely filtered, whereas the second one contains some information which can be used to extrapolate spectrum beyond the cut-off frequency.

3. Extrapolation method

Extrapolation of the spectrum for frequencies removed by transmission channel is used in the process of signal restoration. Prolate spheroidal wave functions Ψ_i [5,6] used in this process are eigenfunctions of the following equation

$$\mathbf{BD}\Psi_i = \lambda_i \Psi_i, \tag{1}$$

where **D** is an operator limiting the signal domain, **B** is an operator limiting the spectrum domain, and λ_i is a respective eigenvalue.

Important property (called double-orthogonality) of the functions Ψ_i is described by two following equations

$$\int_{-\infty}^{\infty} \Psi_i(\omega)\Psi_j(\omega)d\omega = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (i,j = 0,1,2,\ldots) \tag{2}$$

$$\int_{-\Omega/2}^{\Omega/2} \Psi_i(\omega)\Psi_j(\omega)d\omega = \begin{cases} 0 & i \neq j \\ \lambda_i & i = j \end{cases} \quad (i,j = 0,1,2,\ldots) \tag{3}$$

where $\Omega/2$ is the cut-off frequency of transmission channel represented by the operator **B**.

On the basis of the property described by Eq. (2), the spectrum $F(\omega)$ of domain-limited signal $\mathbf{D}f(t)$ can be extrapolated according to the following equation

$$F(\omega) = \sum_{i=0}^{\infty} a_i \Psi_i^c(\omega), \tag{4}$$

where $c = T\Omega/2$ is the size-bandwidth product, T is the size of signal domain and $\Omega/2$ is the cut-off frequency of transmission channel. Index c in Ψ_i^c denotes the set of eigenfunctions of Eq. (1) for the parameter c. The coefficients a_i can be determined from the known (passed by transmission channel) part $\mathbf{B}F(\omega)$ of a signal spectrum. On the basis of the property described by Eq. (3), a_i can be determined in the following way

$$a_i = \frac{1}{\lambda_i} \int_{-\Omega/2}^{\Omega/2} \mathbf{B}F(\omega)\Psi_i^c(\omega)d\omega. \tag{5}$$

The coefficient a_i has the sense of the level of participation of $\Psi_i^c(\omega)$ in the spectrum $F(\omega)$. The eigenvalues λ_i are responsible for normalization of $\Psi_i^c(\omega)$ in the transmitted band. Figure 2 shows the monotonously ordered eigenvalues λ_i for $c = 40$. For such an ordering, the eigenvalues λ_i are approximately equal to 1 if $i < S$, and they rapidly decrease if $i > S$, where $S = 2c/\pi$ and it is called the Shannon number [10]. In communication theory, this is a number of sampling points spaced by the Nyquist distance $R = 2\pi/\Omega$ and it is interpreted as the structural information amount contained in a domain-limited signal transmitted by a band-limited channel with the cut-off frequency $\Omega/2$. The functions Ψ_i of $i < S$ can almost completely restore of the spectrum in the transmitted band. The functions Ψ_i of $i > S$ allow us to extrapolate the spectrum beyond the cut-off frequency. Plots of four functions Ψ_i in the neighbourhood of the Shannon number are presented in Fig. 3.

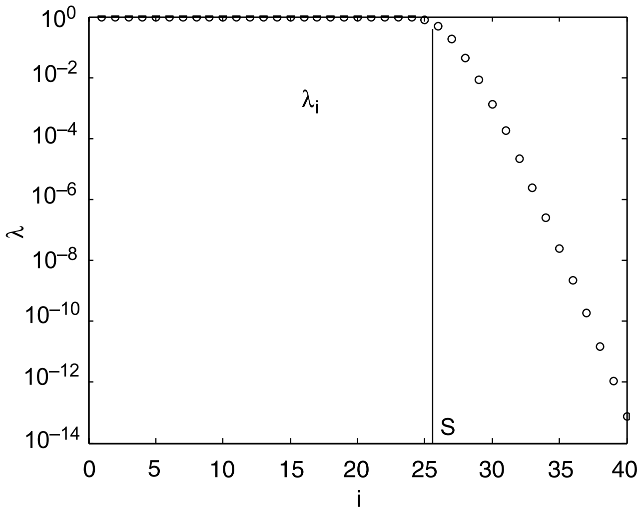

Fig. 2. The monotonously ordered dependence of the eigenvalues λ_i in semi-logarithmic scale for $c = 40$; S is the Shannon number.

4. Realization of the method

The process of spectrum extrapolation is realized according to Eq. (4). Infinity as the upper limit of the sum in this equation guarantees complete reconstruction of the spectrum. In practical realization of the method, infinity has to be replaced there by the finite value N

$$F_{\Omega'}(\omega) = \sum_{i=0}^{N} a_i \Psi_i^c(\omega), \tag{6}$$

where $N + 1$ is the number of the functions Ψ_i which allows us to extrapolate the spectrum up to some frequency value $\Omega'/2$. In this case reconstruction is not complete although improvement in signal resolution should occur. This is possible due to properties of the function Ψ_i. The functions with the lower index have more band-limited spectrum than the functions with the higher index. The eigenfunctions Ψ_i ordered form the lowest to highest index i according to decrease in the value λ_i are used in Eq. (6). Therefore more the functions Ψ_i used in Eq. (6) (greater N) results in extrapolation up to the higher frequency $\Omega'/2$.

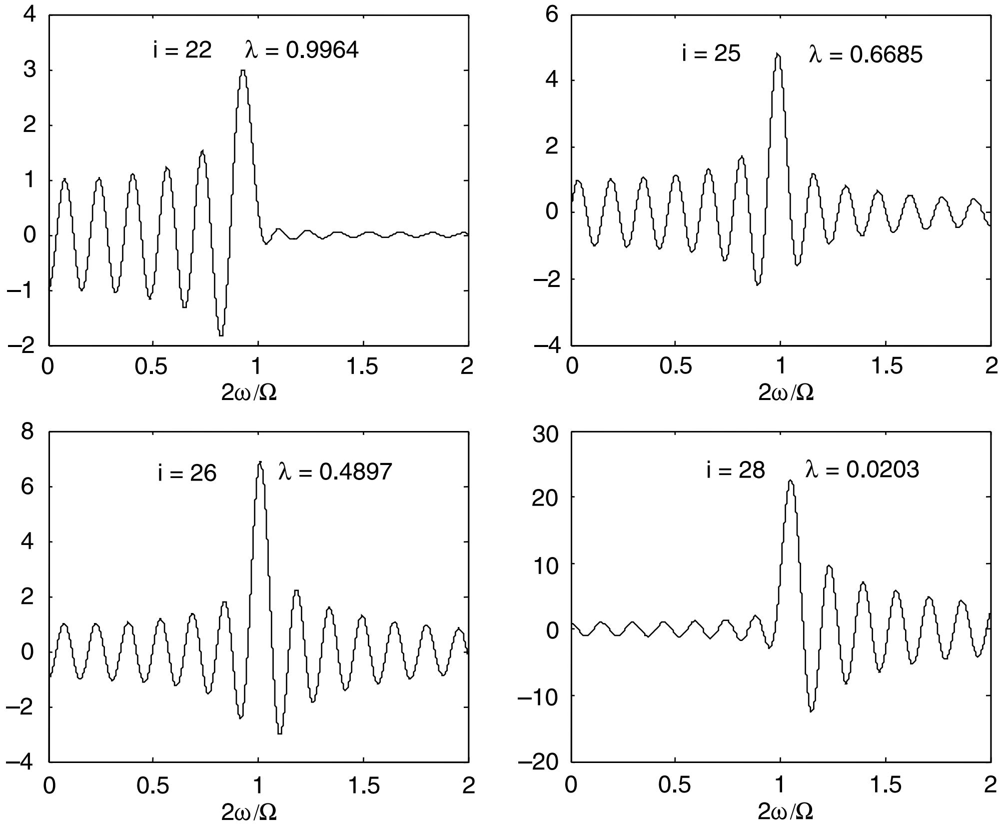

Fig. 3. Plots of four eigenfunctions $\Psi_i^c(\omega)$ in the neighbourhood of the Shannon number for c = 40.

The value $N = S$ allows us to extrapolate the spectrum with $\eta = 1$, where $\eta = \Omega'/\Omega$ and it denotes the extrapolation ratio. In this case, signal improvement cannot be achieved. Out-of-band extrapolation of the spectrum takes place when N is greater then S. The following equation can be used to determine the number N for the desired value of $\Omega'/2$

$$N \approx \frac{\Omega' T}{\pi}. \tag{7}$$

On the other hand, the main limitation in the out-of-band extrapolation is the noise to signal ratio ε/E. The energy part of Ψ_i located in the filtered signal is represented by the square root of the eigenvalue λ_i. The energy of information used for the extrapolation should be greater then the noise energy. For this reason, the following condition should be kept [11]

$$\sqrt{\lambda_i(c)} > \frac{\varepsilon}{E}. \tag{8}$$

The greatest index i for which Eq. (8) is yet satisfied can be used as the number N. Then, in the presence of noise, the extrapolation limit $\Omega'/2$ can be determined from Eq. (7) for such the number N.

As it results from the previous investigation [7], no significant improvement in signal resolution can be achieved when the size-bandwidth product c is too large. This fact can be concluded form behaviour of the prolate spheroidal wave functions. In order to ensure the desired signal-resolution improvement independently of its domain size T, the signal can be divided before transmission into small parts of the domain size $T' < T$ to have the optimal parameter c for extrapolation process. Figure 4 shows the diagram of the signal restoration process according to the mentioned idea of signal fragmentation.

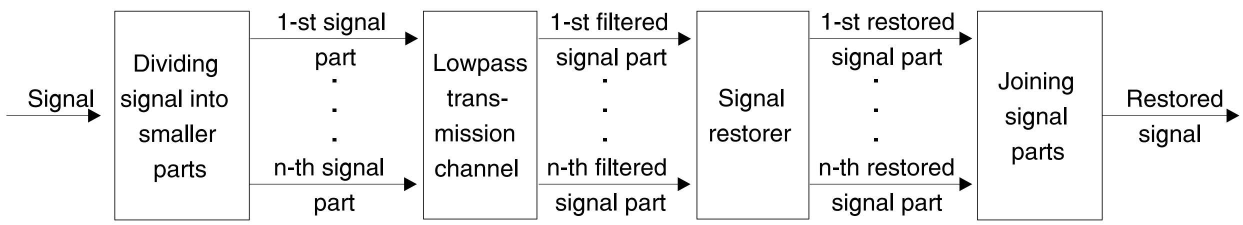

Fig. 4. Diagram of the signal-restoration process by means of signal fragmentation.

Practical realization of the method needs to compute the functions Ψ_i. The prolate spheroidal wave functions $S_{0,i}(c,\xi)$ are applied here in the following way [6]

$$\Psi_i^c(\omega) = \frac{\sqrt{\lambda_i}\, S_{0,i}\left(c, \dfrac{2\omega}{\Omega}\right)}{\sqrt{\displaystyle\int_{-1}^{1} S_{0,i}\left(c, \dfrac{2\omega}{\Omega}\right)^2 d\omega}}. \qquad (9)$$

Computation of the coefficients a_i end extrapolation of the spectrum is realized numerically according to the following equations

$$a_i = \frac{\Omega}{\lambda_i L} \sum_{j=1}^{L} F\left(j\frac{\Omega}{L}\right) \Psi_i^c\left(j\frac{\Omega}{L}\right) \qquad (10)$$

$$F_{\Omega'}\left(j\frac{\Omega}{L}\right) = \sum_{i=0}^{N} a_i \Psi_i^c\left(j\frac{\Omega}{L}\right), \quad j = 1,\ldots,L,\ldots,L\eta \qquad (11)$$

where L is the number of samples taken in discretisation process of the transmitted spectrum band. The following steps should be done in order to realize the presented method:

- determine cut-off frequency $\Omega/2$ of the transmission channel.
- determine desired frequency $\Omega'/2$
- chose the value of parameter c
- compute the size of the signal fragment, $T' = 2c/\Omega$
- determine the value N from Eq. (7)
- determine the noise to signal ratio
- verify condition of Eq. (8)
- divide the signal into fragments of size T'
- transmit signal fragments
- compute the function values $\Psi_i^c\left(j\frac{\Omega}{L}\right)$ for $i = 0, 1, \ldots, N$ and $j = 1, 2, \ldots, \eta L$
- analyse and extrapolate the spectrum of transmitted fragments according Eqs. (10) and (11).

5. Results of numerical experiment

A numerical model has been built to verify the proposed super-resolution method based on signal fragmentation. This model has been implemented in the Matlab 6.0 environment. Original procedures for computing the functions Ψ_i have been also built and used. The original signal fragment $f(t)$ used in the experiment is shown in Fig. 5. This signal consists of two harmonics but only one harmonic can pass through the transmission channel. The higher frequency in the signal is 18% greater then the cut-off frequency $\Omega/2$ of the transmission channel. Therefore the method should ensure the extrapolation ratio $\eta = 1.18$. The other parameters have been assumed as follows: $c = 40$ and $N = 33$. Consequently, the parameter c and the cut-off frequency determine the domain size T' of the signal part resulting from the fragmentation process. The spectrum of such a signal part is presented in Fig. 6. Dark field there denotes the frequencies filtered by the transmission channel.

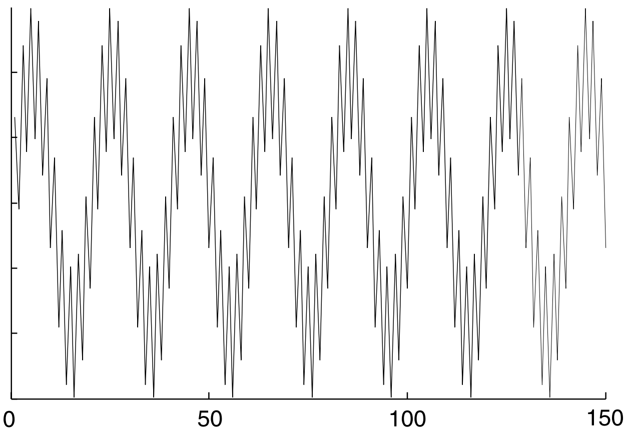

Fig. 5. Signal fragment $f(t)$ used in experiment.

If the original signal is transmitted through the channel, the output signal takes the form presented in Fig. 7. As it can be seen, the higher frequency has been filtered. If the same signal is processed according to the proposed method, described by the diagram in Fig. 4, the output signal takes the form shown in Fig. 8.

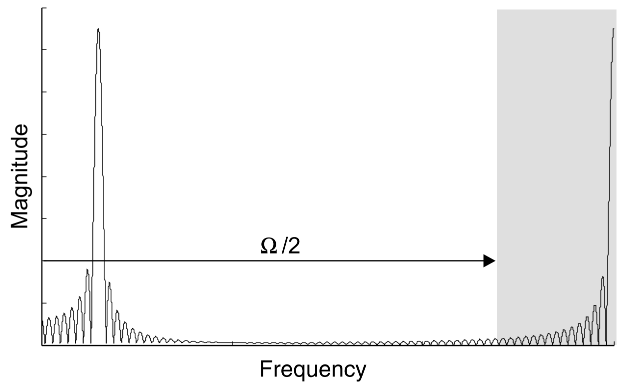

Fig. 6. Spectrum of a signal fragment; dark field denotes the filtered frequencies.

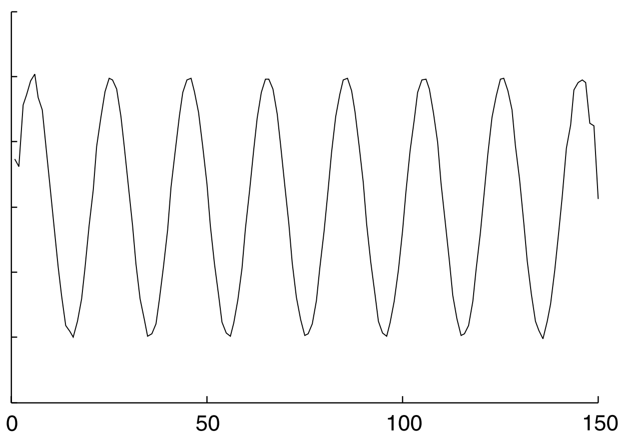

Fig. 7. The output signal for the original signal fragment transmitted through the low-pass channel.

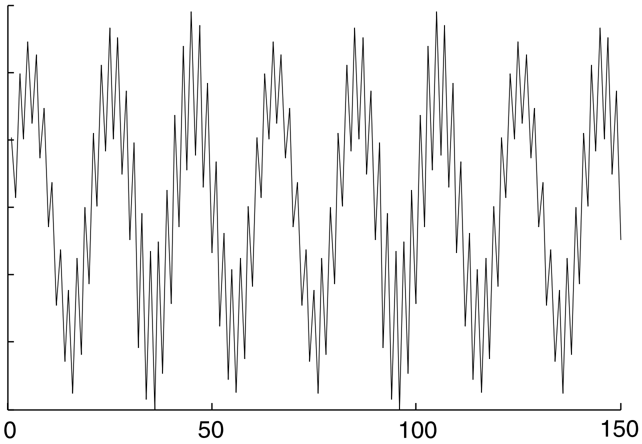

Fig. 8. Signal fragment $\tilde{f}(t)$ restored by means of the proposed method.

An example of restoration of more complex one-dimensional signal is presented in Fig. 9.

Restoration of an image as two-dimensional signal is illustrated in Fig. 10.

The above presented examples of signal restoration were carried without any optimisation. Better results can be obtained by optimising the set of parameter values, more detailed digitalisation of the original signal, and higher computing precision.

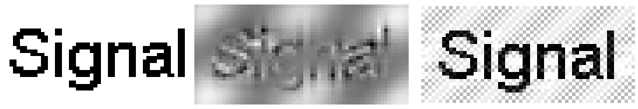

Fig. 10. An example of image restoration. Original, transmitted and restored images are shown at left, middle and right part, respectively.

6. Conclusions

The proposed realization of the super-resolution method allows us to restore signals beyond the Rayleigh resolution limit. The result is independent of the size-bandwidth product. It can be applied to one-dimensional signal like sound and two-dimensional one like image as well. Further research should be focused on optimising the parameter values for the realization method.

An optimisation criterion can be represented by measure of correlation of original and restored signals. Finding parameter values for which the measure reaches maximum could be the aim of optimisation. This goal can be achieved on analytical way as well as numerical simulations.

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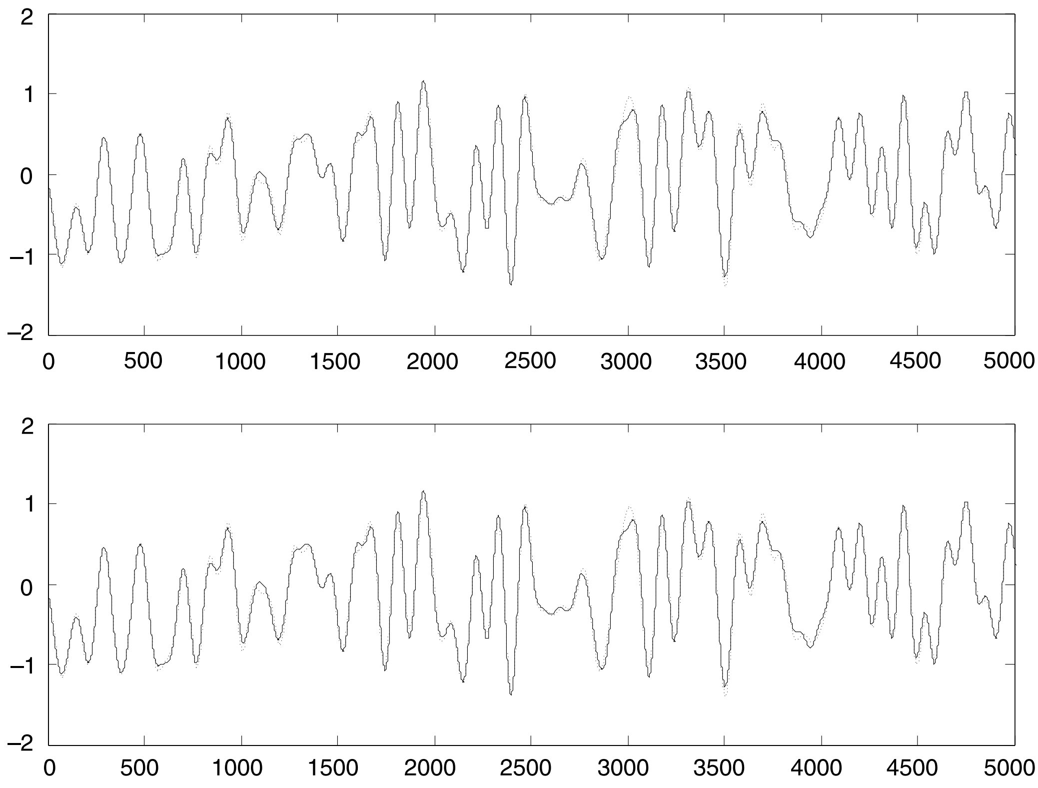

Fig. 9. The signal after transmission through a low-pass channel (top plot) and the restored signal (bottom plot) are compared with the original signal (dotted lines).

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