

# Stable spinning optical solitons in two and three dimensions

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*A brief overview of recent results in the field of multidimensional spinning (vortex) optical solitons is given. We address the stability problem of two- and three-dimensional spinning solitons in media with competing nonlinearities. We conclude that these solitons could be stable, provided that their external size and power (energy) are large enough. The stability of vorticity-carrying solitons is a generic feature of media with competing self-focusing and self-defocusing nonlinearities.*

**Keywords:** optical vortices, spinning optical solitons.

## 1. Introduction

The optics of beams possessing phase singularities has become a very active research area in the last decade (for a brief overview of nonlinear singular optics see, e.g. Ref. 1).

Optical vortices are lines of phase singularity, that is, nodal lines where the intensity of the light, represented by a complex scalar field, vanishes. These wavefront screw dislocations embedded in a host light beam behave like charged particles: they may rotate around the beam axis, repel or attract each other, or annihilate when colliding. Therefore the optical vortex beams have attracted much attention in the last years because of possible applications to all-optical processing of information (for a recent review of theoretical and experimental results in this field, see Ref. 2). Other promising applications of optical vortices are trapping and channeling of matter waves, as well as capture and controlled transport of microparticles, which are trapped by the dark (empty) core of the vortex beam [3], or absorption of particles in spinning motion (by transfer of the angular momentum from the beam) [4].

Very recently, formation of stable bright vortex solitons (spatially localized vortex beams) was observed experimentally, for the first time, in anisotropic photorefractive media through self-trapping of partially incoherent light carrying a phase dislocation [5]; experiments with unstable vorticity-carrying localized beams were reported in both  $\chi^{(2)}$  crystals [6] and in media with a saturable nonlinearity (a hot, dense sodium vapor) [7].

In fact, stability is a major concern for bright vortex solitons, as, unlike their zero-spin counterparts, they are prone to instability against azimuthal perturbations breaking the axial symmetry. A general problem in the study of the azimuthal instability is that it is an oscillatory one, i.e., the associated eigenvalues are complex, and the instability cannot be predicted by dint of known general principles,

such as the Vakhitov-Kolokolov stability criterion, which applies to the case of real instability eigenvalues. In two-dimensional (2D) and three-dimensional (3D) models with quadratic nonlinearities, this instability was discovered in simulations [8–10], and observed in the above-mentioned experiment in the 2D case [6]. As a result, a vortex (spinning) soliton with vorticity  $S = 1$  splits into three or two fragments in the form of separating zero-spin solitons, so that the initial intrinsic spin momentum is transformed into the orbital momentum. Nevertheless, the  $\chi^{(2)}$  nonlinearity acting in combination with the self-defocusing Kerr, nonlinearity gave rise to the first examples of stable spinning (ring-shaped) 2D solitons with  $S = 1$  and  $S = 2$  [11]. The stability of the spinning solitons in this model may be realized as a result of competition between the self-focusing and self-defocusing nonlinearities. This understanding is further supported by the fact that similar stable spinning solitons have also been found in another optical model featuring the competition between focusing and defocusing nonlinearities, viz., the one based on the cubic-quintic (CQ) nonlinear Schrödinger equation [12–18]. An important issue is to identify values of the spin  $S$  at which the 2D vortex solitons may be stable. In the CQ model, only stable solitons with  $S = 1$  were originally identified [12]. Then, it was found that  $S = 2$  vortices have their stability region too [13–15]. Finally, it was demonstrated that the vortex solitons in the CQ model may be stable with the values of spin up to  $S = 5$  [16]; quite plausibly, very narrow stability regions exist for any value of  $S$  in the CQ model. On the other hand, in the  $\chi^{(2)}; \chi^{(3)}$  model stable spinning solitons were thus far found only for  $S = 1$  and  $S = 2$  [11], which suggests to seek for stable solitons with  $S > 2$  in this model too [19].

The most important aspect of the problem is to understand whether the existence of stable spinning solitons with higher values of  $S$  is a peculiarity of the CQ model, or a generic feature [20]. Recently, we have developed a qualitative explanation for the stability of vortex solitons in media with

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competing quadratic and cubic self-defocusing nonlinearities [19]. A remaining challenge is the possible stability of three-dimensional spinning solitons with  $S > 1$  in media with competing nonlinearities; recall that three-dimensional spinning solitons (vortex tori) with topological charge  $S = 1$  have their stability region in such models [21–23]. A noteworthy corollary of the stability results for  $S > 1$  is that, in the models with competing nonlinearities, dark solitons with multiple values of the topological charge, which may be considered as a limiting case of the bright ones with an infinitely large size, are stable too (while in the well-known model with the self-defocusing  $\chi_{-}^{(3)}$  nonlinearity, all dark vortices with  $S > 1$  are unstable [24]).

However, in media with competing nonlinearities, robust soliton complexes (in the form of “clusters” or soliton “molecules”) composed by several nonspinning solitons were introduced, too [25–28]. It was found that the quasi-stable propagation of such soliton clusters is a generic feature of media with competing nonlinearities (self-focusing cubic and self-defocusing quintic nonlinearities or quadratic nonlinearities in competition with self-defocusing cubic nonlinearities).

This paper is organized as follows. In the next section, we present simulations showing the stability of spinning solitons and their resistance to the input random noise. Also, a qualitative explanation of the stability of the vortex (spinning) solitons, including the fact that the stability region exponentially shrinks with the increase of  $S$  is given. In the last section we briefly present the conclusions.

## 2. Spinning solitons

Next, we restrict ourselves to equations describing the  $\chi^{(2)}$  coupling between the fundamental-frequency (FF) and second-harmonic (SH) fields  $u$  and  $v$  in the presence of the self-defocusing  $\chi^{(3)}$  nonlinearity in the (2+1)-dimensional geometry are well known [11,29–31]:

$$\begin{aligned} i \frac{\partial u}{\partial Z} + \frac{1}{2} \nabla^2 u + u^* v - (|u|^2 + 2|v|^2)u &= 0, \\ i \frac{\partial v}{\partial Z} + \frac{1}{2} \nabla^2 v - \beta v + u^2 - 2(|u|^2 + |v|^2)v &= 0. \end{aligned} \quad (1)$$

Here,  $*$  stands for the complex conjugation,  $Z$  is the propagation distance,  $\nabla^2$  is the diffraction operator acting on the normalized transverse spatial coordinates  $X$  and  $Y$ , and  $\beta$  is a phase-mismatch parameter. Equations (1) assume that the Poynting-vector walk-off between the harmonics is compensated [31–32].

We look for stationary solutions to Eqs. (1) in the form of

$$u = U(r)\exp(i\kappa Z + iS\theta), \quad v = V(r)\exp[2(i\kappa Z + iS\theta)],$$

where  $(r, \theta)$  are the polar coordinates in the plane  $(X, Y)$ ,  $\kappa$  is the wave number, and the integer  $S$  is the above-mentioned spin. The amplitudes  $U$  and  $V$  may be assumed real, obeying the equations

$$\begin{aligned} (U'' + r^{-1}U' - S^2 r^{-2}U) \\ - 2[\kappa U - UV + (U^2 + 2V^2)U] &= 0, \\ (V'' + r^{-1}V' - 4S^2 r^{-2}V) \\ - 4[(2\kappa + \beta)V - U^2 + 2(U^2 + V^2)V] &= 0, \end{aligned} \quad (2)$$

where the prime stands for  $d/dr$ .

The dynamical Eqs. (1) conserve the total energy (norm), the Hamiltonian, momentum (equal to zero for the solutions considered), and angular momentum in the transverse plane.

We have numerically found one-parameter families of stationary 2D spinning solitons having a ring-like shape with a hole in its center, which is supported by the phase dislocation. To this aim, we solved Eqs. (2) using the standard band-matrix algorithm to deal with the corresponding two-point boundary-value problem.

Obviously, the wave number  $\kappa$  must exceed the cutoff value,  $\kappa \geq \kappa_{\text{cutoff}} \equiv \max(0, -\beta/2)$ , for the fields to be exponentially localized. For a fixed mismatch  $\beta$ , the stationary zero-spin and spinning solitons exist in a limited region, with  $\kappa$  ranging from  $\kappa_{\text{cutoff}}$  up to a certain upper limit (offset value)  $\kappa_{\text{offset}}$ , at which the soliton's power diverges due to the divergence of its outer radius  $R$ , while the field amplitudes  $U$  and  $V$  remain finite.

Further straightforward analysis of Eqs. (2) shows that  $R$  diverges logarithmically at  $\kappa_{\text{offset}} - \kappa \rightarrow 0$ ,

$$R \sim \ln[1/(\kappa_{\text{offset}} - \kappa)] \quad (3)$$

The stability results are summarized in Fig. 1, where the continuous lines border the existence domain of the localized ring-shaped solitons, and the dashed lines are boundaries between stable and unstable regions in the parameter plane  $(\beta, \kappa)$ . In regions A and D in Fig. 1, no localized solutions exist: in region A – because the wave number  $\kappa$  is below the cutoff, and in region D – because the defocusing  $\chi^{(3)}$  nonlinearity becomes dominant, preventing the formation of solitons. For  $\beta < 0$ , there is a narrow strip (region B in Fig. 1) where the zero-spin solitons exist but are unstable. Stable zero-spinning and unstable spinning solitons coexist in region C. Stable spinning solitons with  $S = 1$ ,  $S = 2$ , and  $S \geq 3$  exist in small domains near the offset line (the continuous line separating regions C and D), the dashed lines located near the offset line being boundaries of the stability regions for the spinning solitons. We have found that, regardless of the value of the phase mismatch, the spinning solitons with  $S = 3$  and  $S = 4$  are stable in regions occupying  $\cong 3\%$ , respectively  $\cong 1.5\%$  of their existence domain.

However, one can assume that, very generally speaking, the spinning soliton is not an absolutely stable object, but rather a metastable one. Indeed, the energy of the spinning soliton is larger than that of its zero-spin counterpart, hence it might be possible that a very strong initial perturbation will provoke its rearrangement into a zero-spin

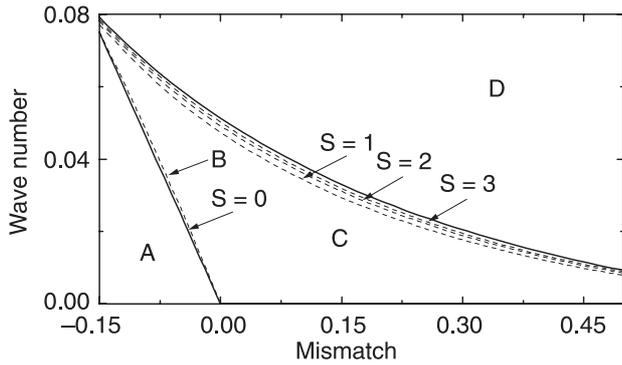


Fig. 1. Existence and stability domains for bright fundamental and vortex solitons with  $S = 0, 1, 2,$  and  $3$ . The upper continuous curve separating the domains  $C$  and  $D$  is their common existence border, corresponding to infinitely broad solitons. The stability region of the vortices with  $S = 4$ , which is very narrow, is explained in the text.

soliton, the angular momentum being carried away with emitted radiation. In terms of this consideration, it appears that the  $S = 1$  and  $S = 0$  solitons are separated by extremely high potential barriers, which make the assumed process practically impossible. To illustrate this point, in Fig. 2 we show the cross sections of the  $S = 1$  soliton which was very strongly perturbed at the initial point,  $Z = 0$  (the perturbation is about 30% of the soliton's amplitude), and the result of its evolution at the point  $Z = 200$ . For the same case, the comparison of the distributions of the intensity and phase inside the initial strongly perturbed soliton and the finally established one are shown in Fig. 3. As is obvious from Figs. 2 and 3, the soliton was able to completely heal the damage, remaining a truly stable object.

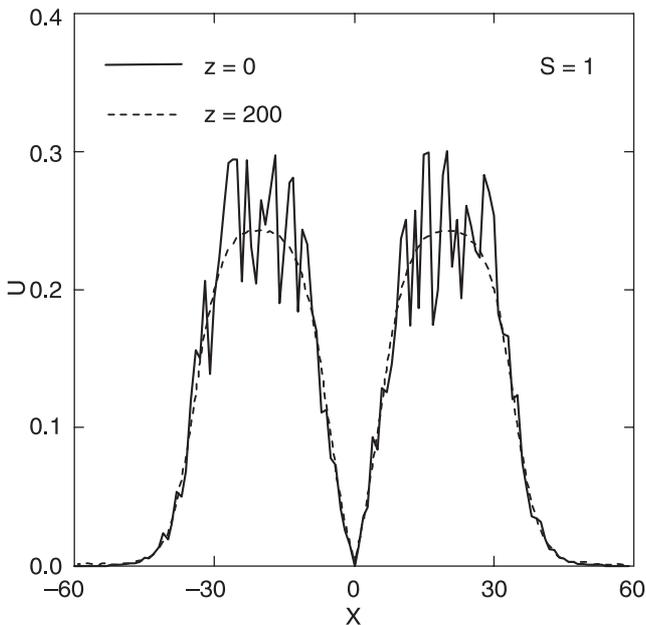


Fig. 2. Cross sections of an  $S = 1$  soliton that was strongly perturbed at  $Z = 0$ , and the result of its evolution after having passed the propagation distance  $Z = 200$ .

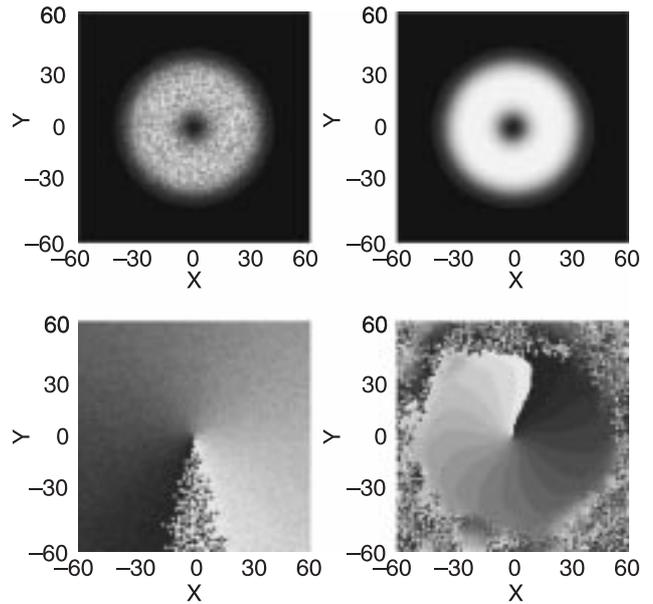


Fig. 3. The recovery of the soliton with the spin  $S = 1$  in the same case as in Fig. 2.

A simpler but more general analysis may be based on considering the vortex soliton as a two-dimensional “liquid drop” of the annular shape, with inner and outer borders (for more details see Ref. 19). Then, an obvious stability criterion is the minimization of the “surface tension”, i.e., of the total length (perimeter) of the borders (the total area of the drop, or of a set of secondary drops into which the original unstable one may split, is approximately conserved due to the power conservation). The outer and inner radii of the annulus being  $R$  [see Eq. (3)] and  $\rho$ , its area and perimeter are  $S = \pi(R^2 - \rho^2)$ ,  $L = 2\pi(R + \rho)$ . If the annular drop is unstable against splitting into  $n$  round-shaped ones with the radius  $\tilde{R}$ , the area conservation yields  $\tilde{R} = \sqrt{S/(\pi n)}$ . Accordingly, the total perimeter of the set of the secondary drops is  $l \equiv 2\pi\tilde{R}n$ . As it follows from these equations, the ratio of the perimeters of the split and unsplit configurations is  $l/L = \sqrt{n(R - \rho)/(R + \rho)}$ .

An obvious consequence of the above estimate is that the condition  $l/L > 1$ , which implies absolute stability of the annulus against the splitting into  $n$  drops, is  $R > (n + 1)\rho/(n - 1)$ . The strongest condition following this relation corresponds to  $n = 2$  (recall that the exact numerical results demonstrate that the instability mode with the azimuthal index  $n = 2$ , which implies the beginning of the splitting into two fragments, is indeed the most persistent one), giving the estimate  $R > 3\rho$ . For  $\kappa$  sufficiently close to  $\kappa_{\text{offset}}$ ,  $\rho$  depends only on the vorticity  $S$  of the annular soliton, while  $R$  may be indefinitely large, depending on the soliton's power. Thus, the above condition predicts that the vortex solitons of a sufficiently large size may indeed be stable against the splitting.

Further, to estimate the dependence of the stability region on  $S$ , we may use a crude estimate for  $\rho$ , following

from the matching of the asymptotic form of the solution valid for  $\rho \rightarrow 0$ . This procedure predicts the following dependence of the hole's radius on large values of  $S$ ,

$$\rho = \rho_0 \sqrt{S}, \quad (4)$$

with the constant  $\rho_0$ . The fact that the model must be based on competing self-focusing and self-defocusing nonlinearities implicitly comes into the play at this stage of the consideration, as otherwise the quasi-flat field in the inner region of the annulus would be subject to the modulational instability.

Finally, we predict that the relative size of the stability region for the vortex solitons shrinks, with the increase of  $S$ , as

$$\Delta\kappa/\kappa_{\text{offset}} \sim \exp(-C_2 \sqrt{S}), \quad (5)$$

with a certain constant  $C_2$ , depending on the dimension of the model ( $D = 2$  in our case), see Ref. 19. In fact, the prediction of Eq. (5) is universal (model-independent), while  $C_2$  may depend on parameters of the particular model, such the mismatch constant  $\beta$  in the present model with competing quadratic and quintic nonlinearities. Note that a similar analysis for the case of three-dimensional spinning solitons gives us an estimate of the stability region which also shrinks exponentially as  $(-C_3 \sqrt{S})$ .

### 3. Conclusions

In this work, we have briefly reviewed the problem of stability of vortex solitons in two-dimensional media combining competing nonlinearities. The model describes propagation of localized beams with intrinsic vorticity  $S$  in the bulk optical medium. An earlier established result was that the vortex solitons with  $S = 1$  and  $S = 2$  could be stable, provided that their external size and power are large enough, but it was assumed that all the higher-order solitons with  $S \geq 3$  would be unstable. In contrast with this, it has recently been found that in another model, with the cubic-quintic nonlinearity, solitons with  $S > 2$  had their (narrow) stability regions too. We have demonstrated that the same is true in the  $\chi^{(2)}$ : $\chi^{(3)}$  model too. In particular, the  $S = 3$  and  $S = 4$  solitons are stable in regions which occupy, respectively,  $\cong 3\%$  and  $1.5\%$  of their existence domain [19]. Solitons with still larger  $S$  also have very narrow stability regions. These results were obtained by means of calculation of the stability eigenvalues, and checked in direct simulations. It has also been demonstrated that the stable solitons are truly robust, readily self-trapping from a rather arbitrary initial beam with the embedded vorticity, and easily cleaning themselves from large random perturbations. Besides the numerical results, we have also proposed a simple qualitative explanation for the stability of the broad vortices against splitting into a set of zero-spin solitons. In particular, this analysis predicts that, for large  $S$ , the width of stability region shrinks exponentially as

$\exp(-C\sqrt{S})$ , where  $C$  is a constant depending on the concrete model and on the dimension  $D$ . Thus, we conclude that the stability of higher-order spinning solitons is a generic feature of optical media with competing self-focusing and self-defocusing nonlinearities.

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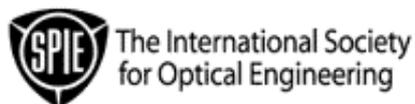
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